

$$7a) \quad y dy = -x dx \Rightarrow \int y dy = - \int x dx \Rightarrow \frac{1}{2} y^2 = -\frac{1}{2} x^2 + C_1$$

$$\Rightarrow y = \pm \sqrt{2C - x^2} \quad (4P)$$

$$y(0) = -2 \Rightarrow \frac{1}{2}(-2)^2 = -\frac{1}{2}0^2 + C_1 \Rightarrow C_1 = 2 \quad (4P)$$

$$b) \quad \frac{y}{\sqrt{1+y^2}} dy = dx \Rightarrow \int \frac{y dy}{\sqrt{1+y^2}} = \int dx \Rightarrow \frac{1}{2} \ln(1+y^2) = x + C_1$$

$$\Rightarrow 1+y^2 = e^{2C+2x} = A e^{2x} \Rightarrow y = \pm \sqrt{A e^{2x} - 1} \quad (4P)$$

$$y(0) = 1 \Rightarrow 1+0^2 = A e^0 \Rightarrow A = 1 \quad (4P)$$

$$c) \quad \frac{dy}{1+y^2} = e^x dx \Rightarrow \int \frac{dy}{1+y^2} = \int e^x dx \Rightarrow \arctan(y) = e^x + C$$

$$\Rightarrow y = \tan(e^x + C_1) \quad (4P)$$

$$y(0) = -1 \Rightarrow \arctan(-1) = e^0 + C_1 \Rightarrow C_1 = -\frac{\pi}{4} - 1 \quad (4P)$$

$$d) \quad y' = (e^x + 1)(y-2) \Rightarrow \frac{dy}{y-2} = (e^x + 1) dx$$

$$\Rightarrow \int \frac{dy}{y-2} = \int (e^x + 1) dx \Rightarrow \ln|y-2| = e^x + x + C_1 \quad (4P)$$

$$\Rightarrow y-2 = e^{e^x+x+C_1} = A e^{(e^x+x)} \Rightarrow y = A e^{(e^x+x)} + 2 \quad (4P)$$

$$y(0) = 0 \Rightarrow 0 = A e^{1+0} + 2 \Rightarrow A = -\frac{2}{e} \quad (4P)$$

$$e) \quad \frac{dy}{y} = \frac{x}{x-1} dx = \left(1 + \frac{1}{x-1}\right) dx \Rightarrow \int \frac{dy}{y} = \int \left(1 + \frac{1}{x-1}\right) dx$$

$$\Rightarrow \ln|y| = x + \ln|x-1| + C_1 \quad (4P)$$

$$\Rightarrow y = e^{x+C_1} (x-1) = A e^x (x-1) \quad (4P)$$

$$y(0) = 2 \Rightarrow 2 = A e^0 (-1) \Rightarrow A = -2 \quad (4P)$$

$$8a) \quad y' = (xz)' = xz' + z$$

$$\Rightarrow xz' + z = -\frac{xz}{x} - \frac{x}{xz} = -z - \frac{1}{z}$$

$$\Rightarrow xz' = -2z - \frac{1}{z} = -\frac{1+2z^2}{z} \quad (6P)$$

$$b) \quad \frac{z}{1+2z^2} dz = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{4} \ln(1+2z^2) = -\ln|x| + C_1 \quad (4P)$$

$$\Rightarrow 1+2z^2 = \frac{e^{4C_1}}{x^4} = \frac{A}{x^4}$$

$$\Rightarrow z^2 = \frac{A}{2x^4} - \frac{1}{2} \quad (2P)$$

$$\Rightarrow y^2 = x^2 z^2 = \frac{A}{2x^2} - \frac{x^2}{2}$$

$$\Rightarrow y = \pm \sqrt{\frac{A}{2x^2} - \frac{x^2}{2}} \quad (4P)$$

$$g) \quad y' = xz \quad y' = xz' + z \quad (4P)$$

$$\Rightarrow xz' + z = z + e^{-z} \Rightarrow xz' = e^{-z}$$

$$\Rightarrow e^z dz = \frac{dx}{x} \quad (4P)$$

$$\Rightarrow e^z = \ln|x| + C' \quad (4P)$$

$$\Rightarrow z = \ln(\ln|x| + C') \quad (4P)$$

$$\Rightarrow y = x \ln(\ln|x| + C') \quad (4P)$$

$G)$ $y = 1$ (locally) is a solution since

$$y' = 0 = \sqrt{1-y} = \sqrt{y-1}$$

(which is bounded) (20P)

Remark:

Separation of variables yields

$$y = 1 + \left(x \pm C\right)^2$$

with $C' = 0$ if $y(0) = 1$. But the solution

with initial condition $y(0) = 1$ is not unique.

Since $\sqrt{1-y}$ is "not nice" at $y = 1$

(not Lipschitz continuous at $y = 1$).